The expected number of components in subgraphs of a small world network derived from a circulant.

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and

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Circulant:

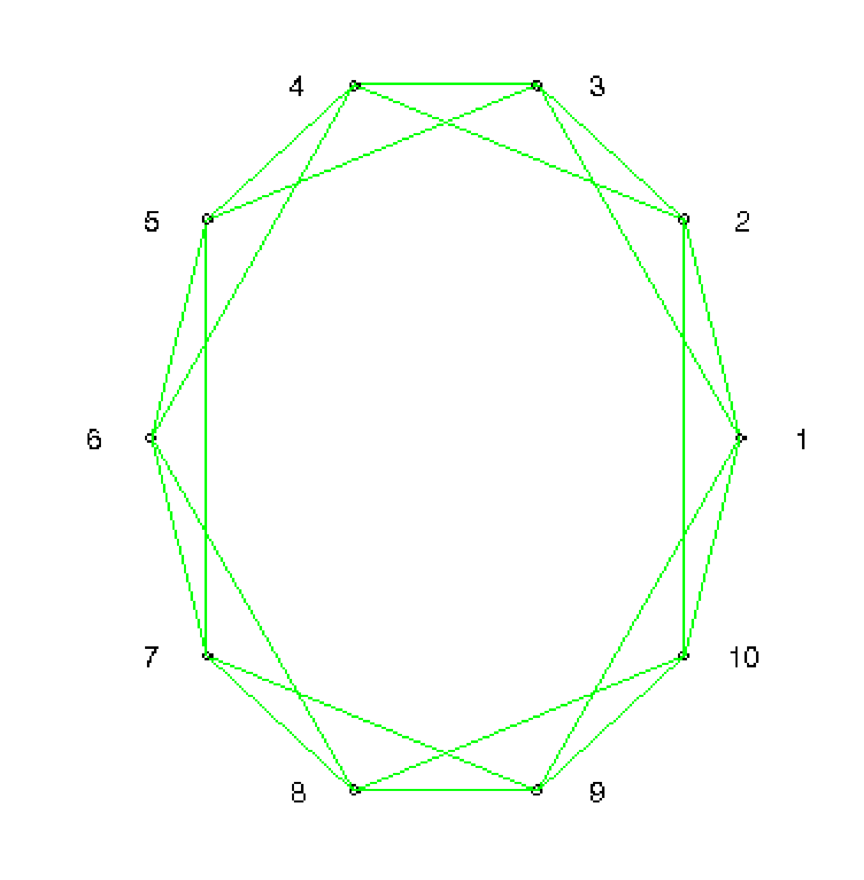
Definition: A circulant graph is a graph of n vertices in which the ith vertex is adjacent to the $(i - k)$th and $(i + k)$th vertices for some k. (This k is referred to as the regularity parameter.)

Adjacency Matrix:

The adjacency matrix for such a graph has 0's on the diagonals and k 1's on each side of these zeros, in each row. In row j, if there are not k places to the left of the diagonal, we place 1's in the last $k - j$ places in row j. Likewise, if there are not k places to the right of the diagonal, we place 1's in the first $k - j$ places in row j.

Example of a 2k regular circulant with k = 2:

Graph:



Matrix:

$$
\begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}
$$

Induced Subgraph:

Definition: Let G = (V, E) be a graph, with V = set of vertices and E = set of edges. Let W be a subset of V and F be a subset of E.

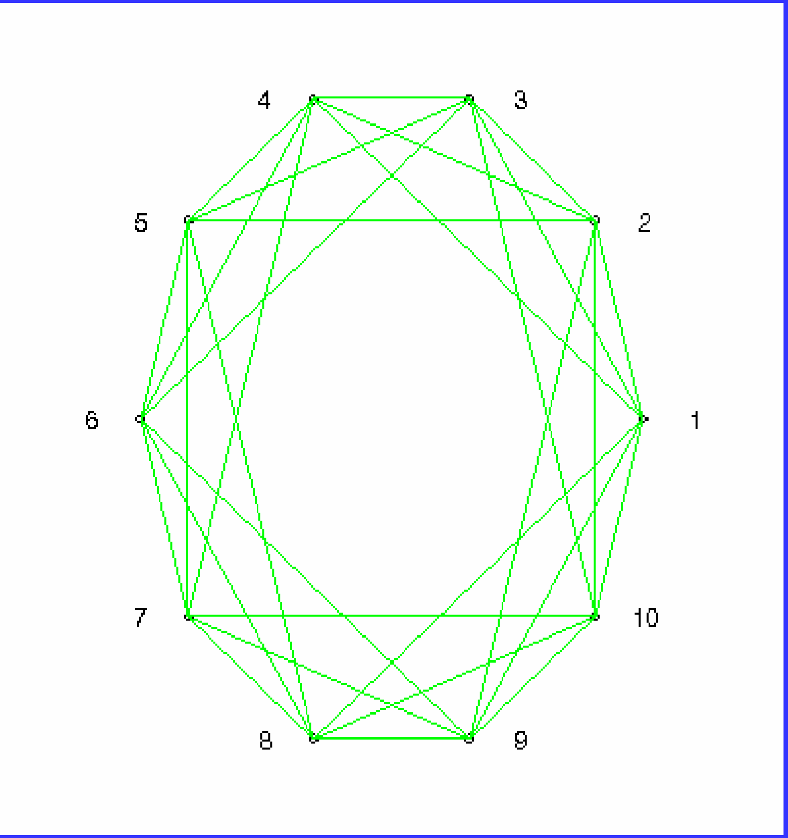
The graph (W, F) is an Induced Subgraph of G if F = the intersection of E and all 2 element subsets of W.

Components:

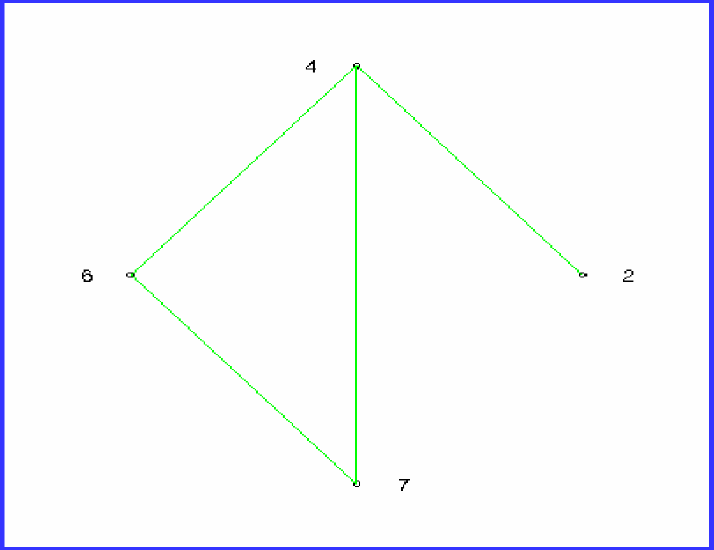
Definition: A component of a graph is a maximal connected subgraph.

Examples of Subgraphs:

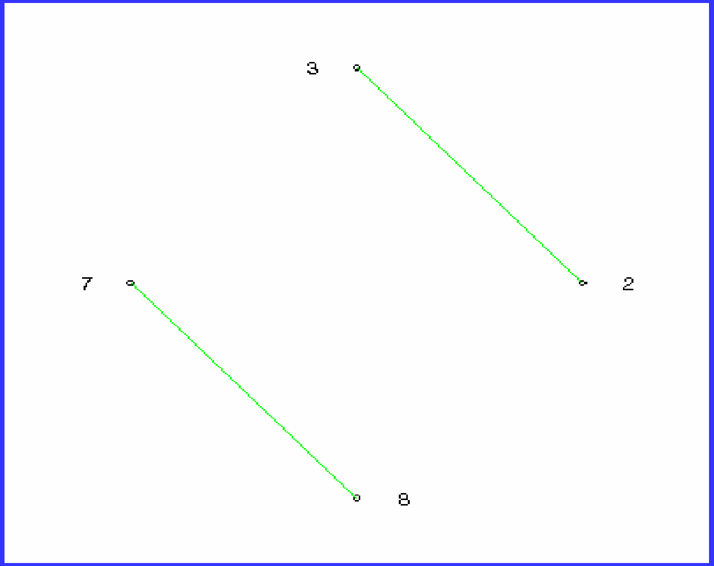
Original Circulant



Subgraph of 1 Component



Subgraph of 2 Components



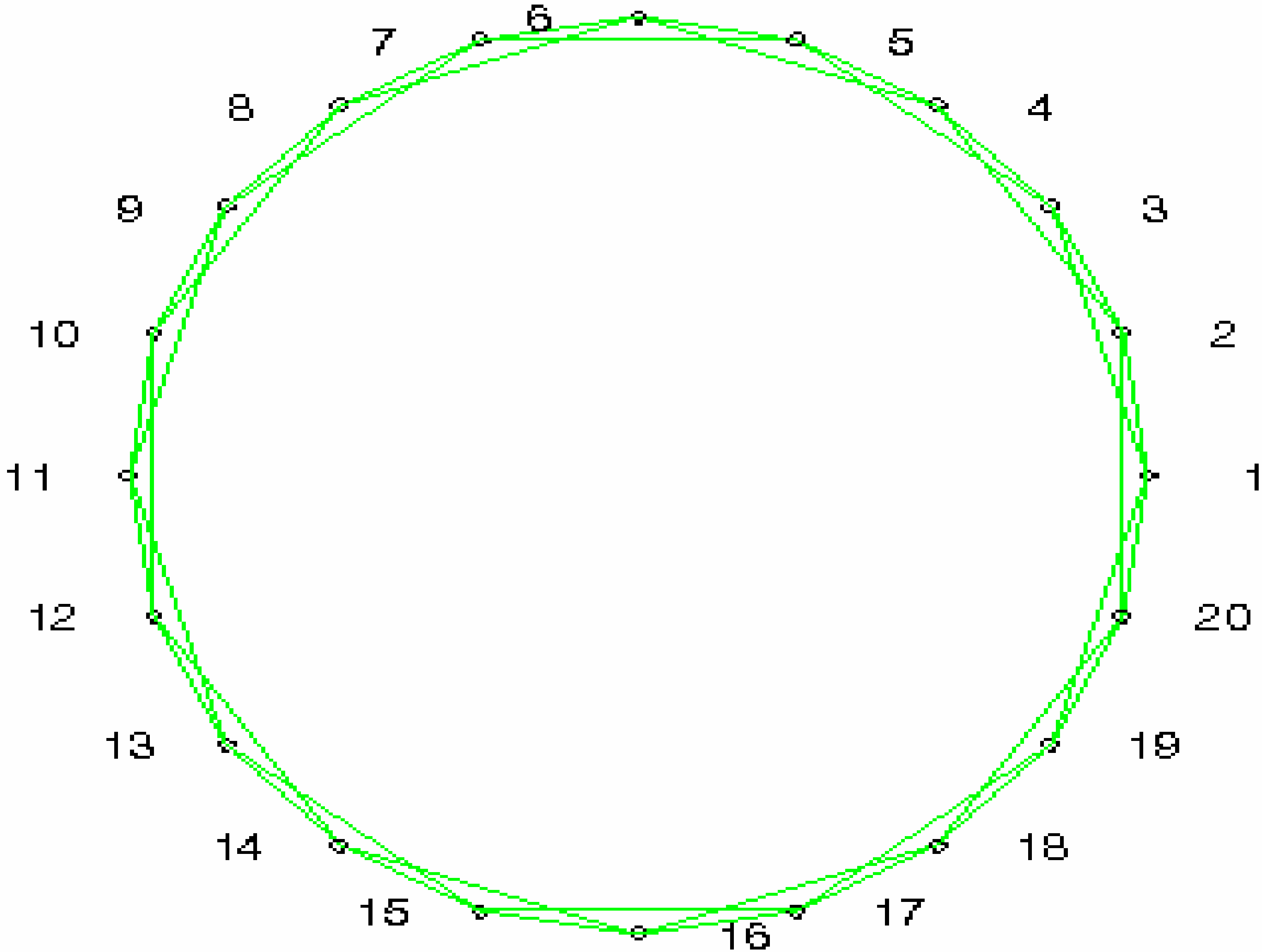
Fundamental Question #1:

What's the *Expected Number* of *Components* in an *Induced Subgraph* of a *Circulant*???

First, lets reduce the problem.....

We can view the vertices of a subgraph as the original vertices being renumbered by 0s or 1s.

We replace a vertex with 1 if it is in the subgraph, and 0 if it is not in the subgraph.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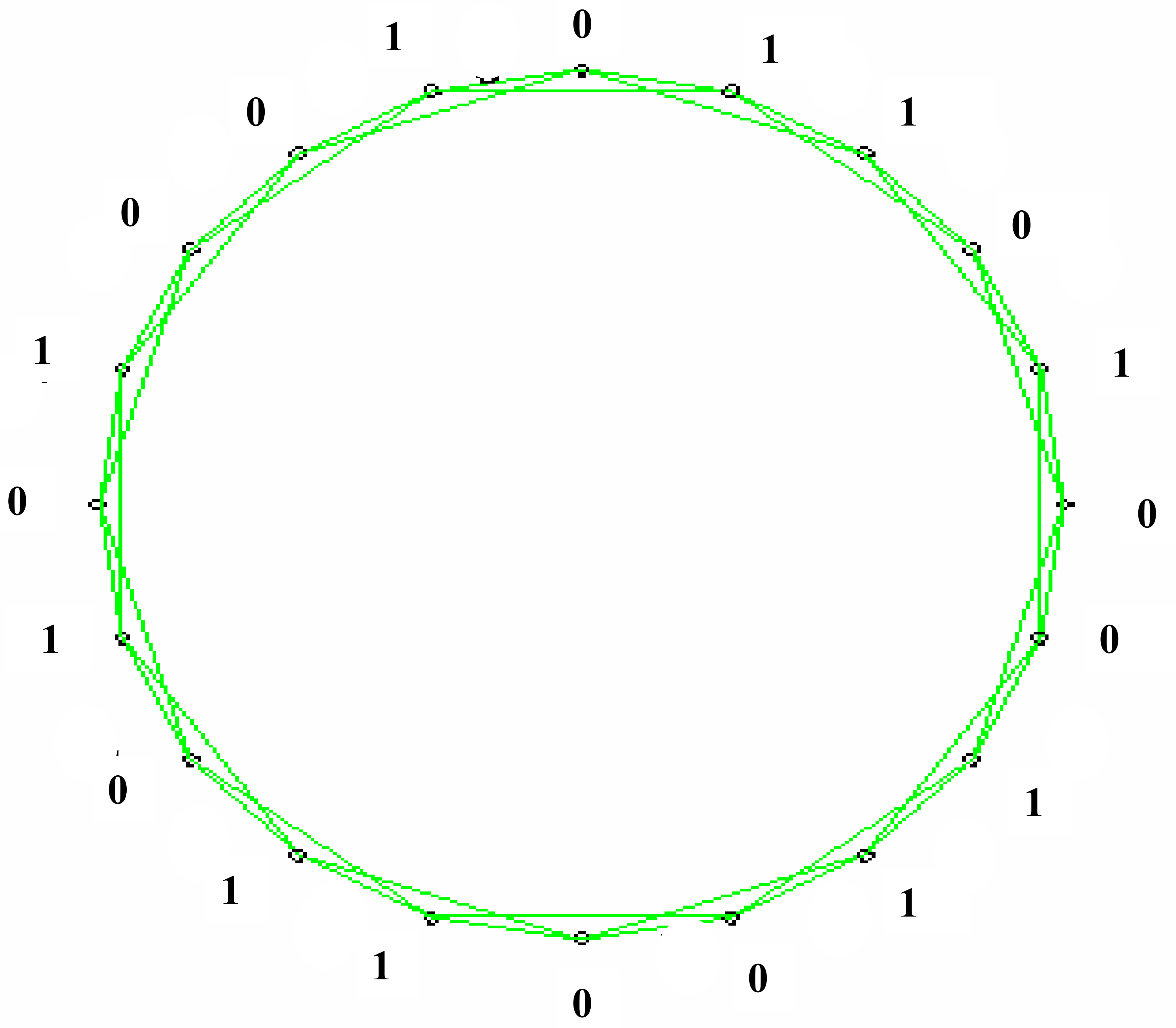
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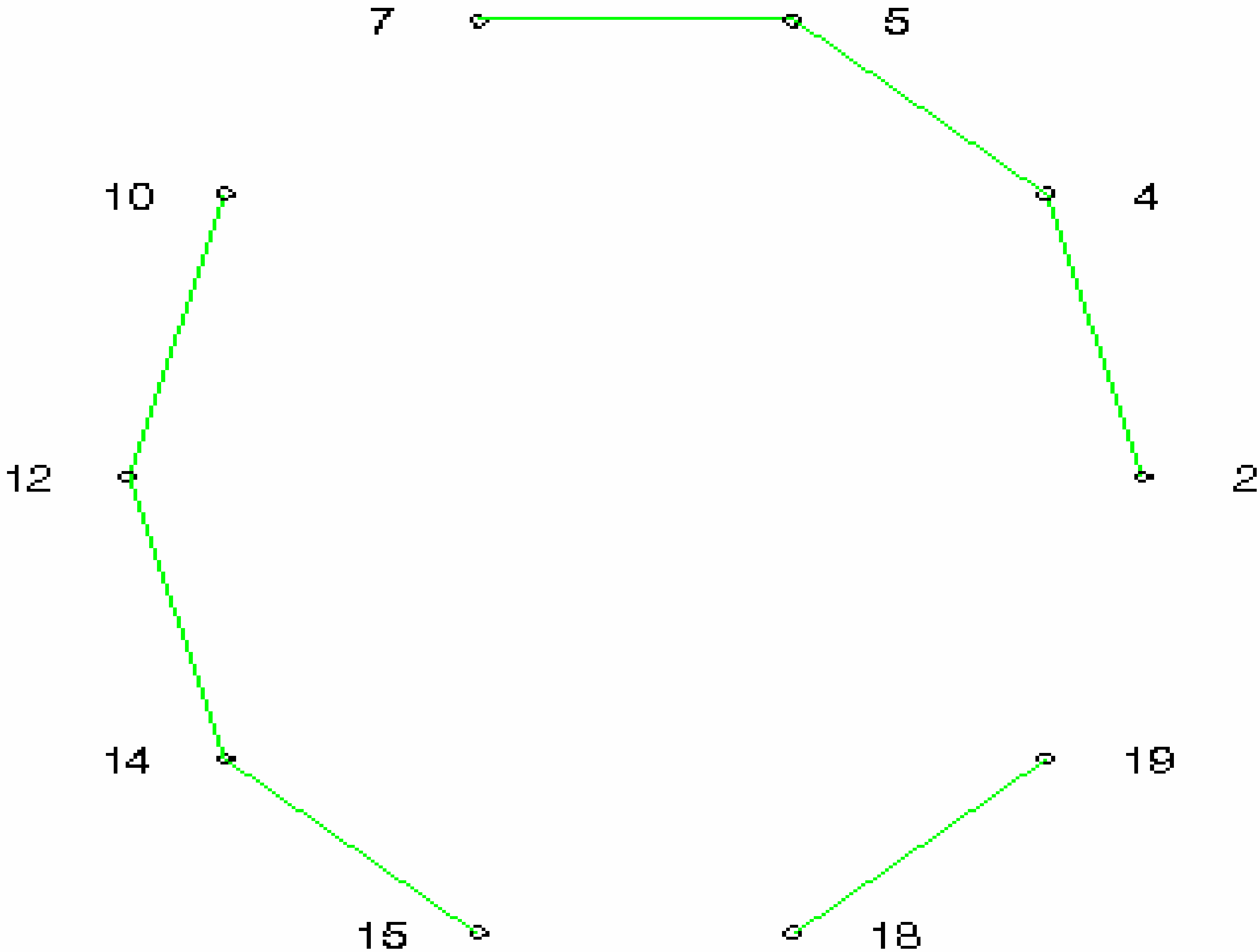
7
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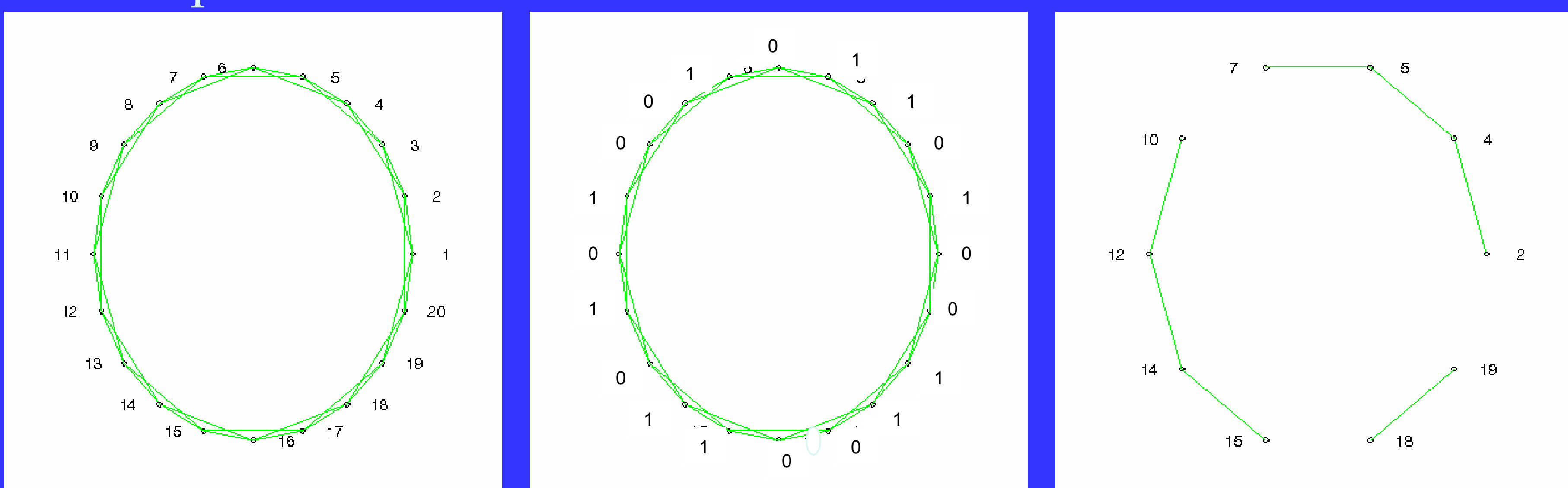
15
18

First, lets reduce the problem.....

We can view the vertices of a subgraph as the original vertices being labeled by 0s or 1s.

We label a vertex with 1 if it is in the subgraph, and 0 if it is not in the subgraph.

Example:



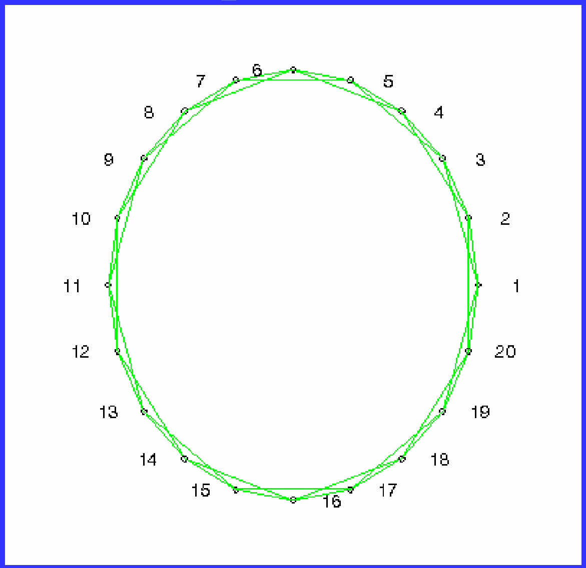
First, lets reduce the problem.....

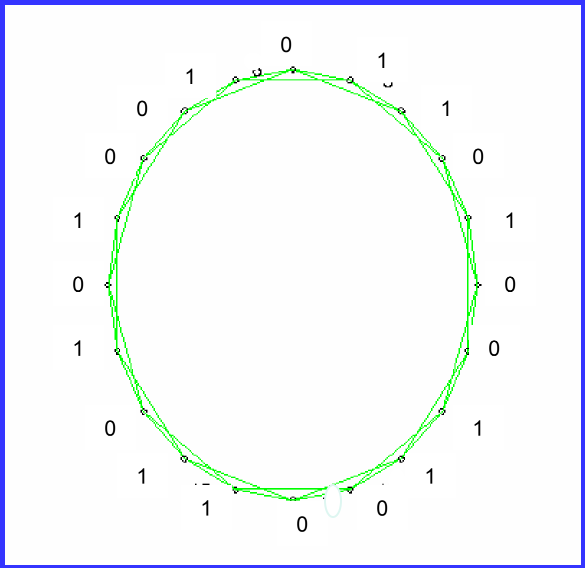
We can view the vertices of a subgraph as the original vertices being labeled by 0s or 1s.

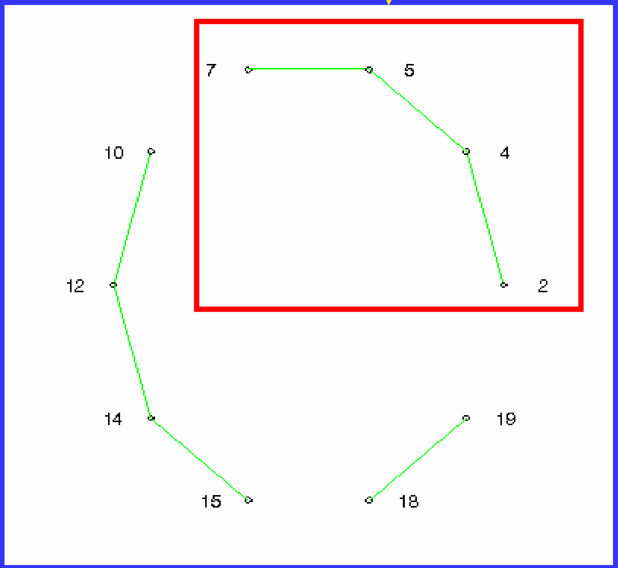
We label a vertex with 1 if it is in the subgraph, and 0 if it is not in the subgraph.

First Component

Example:





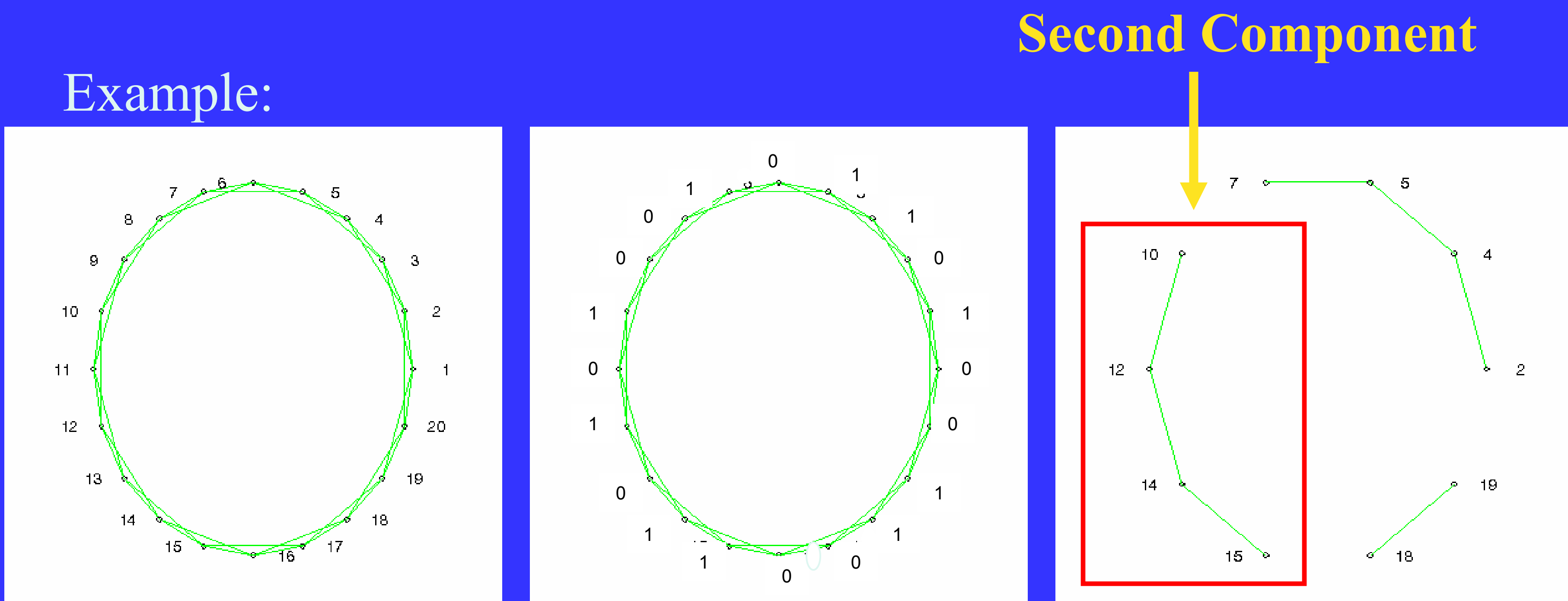


First, lets reduce the problem.....

We can view the vertices of a subgraph as the original vertices being labeled by 0s or 1s.

We label a vertex with 1 if it is in the subgraph, and 0 if it is not in the subgraph.

Example:



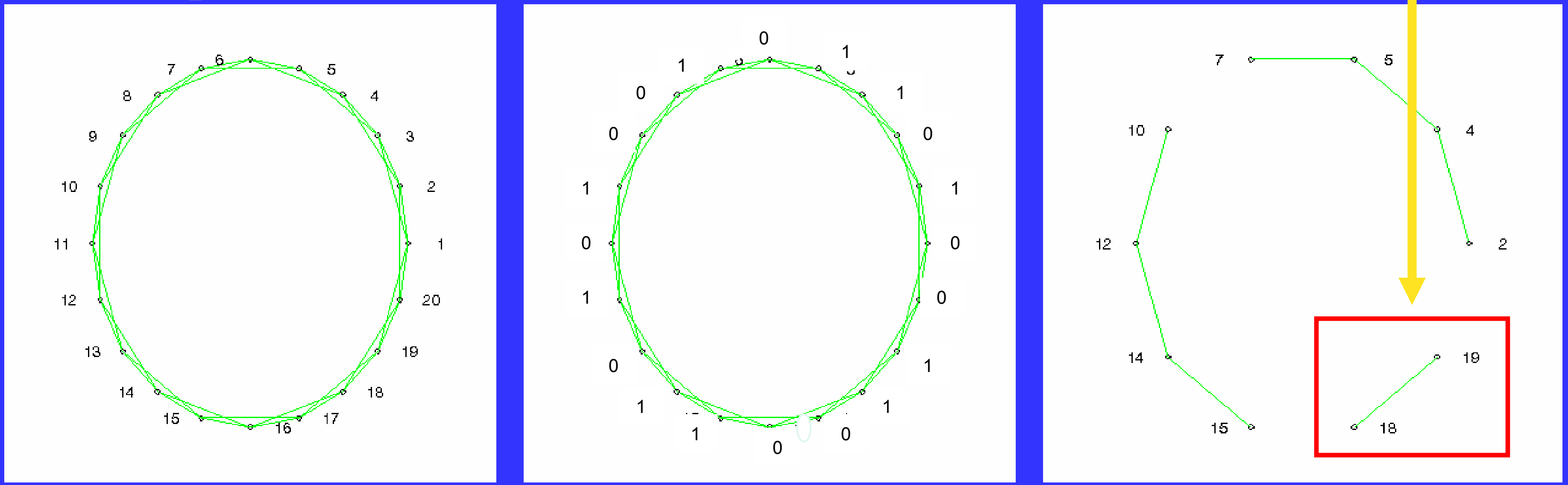
First, lets reduce the problem.....

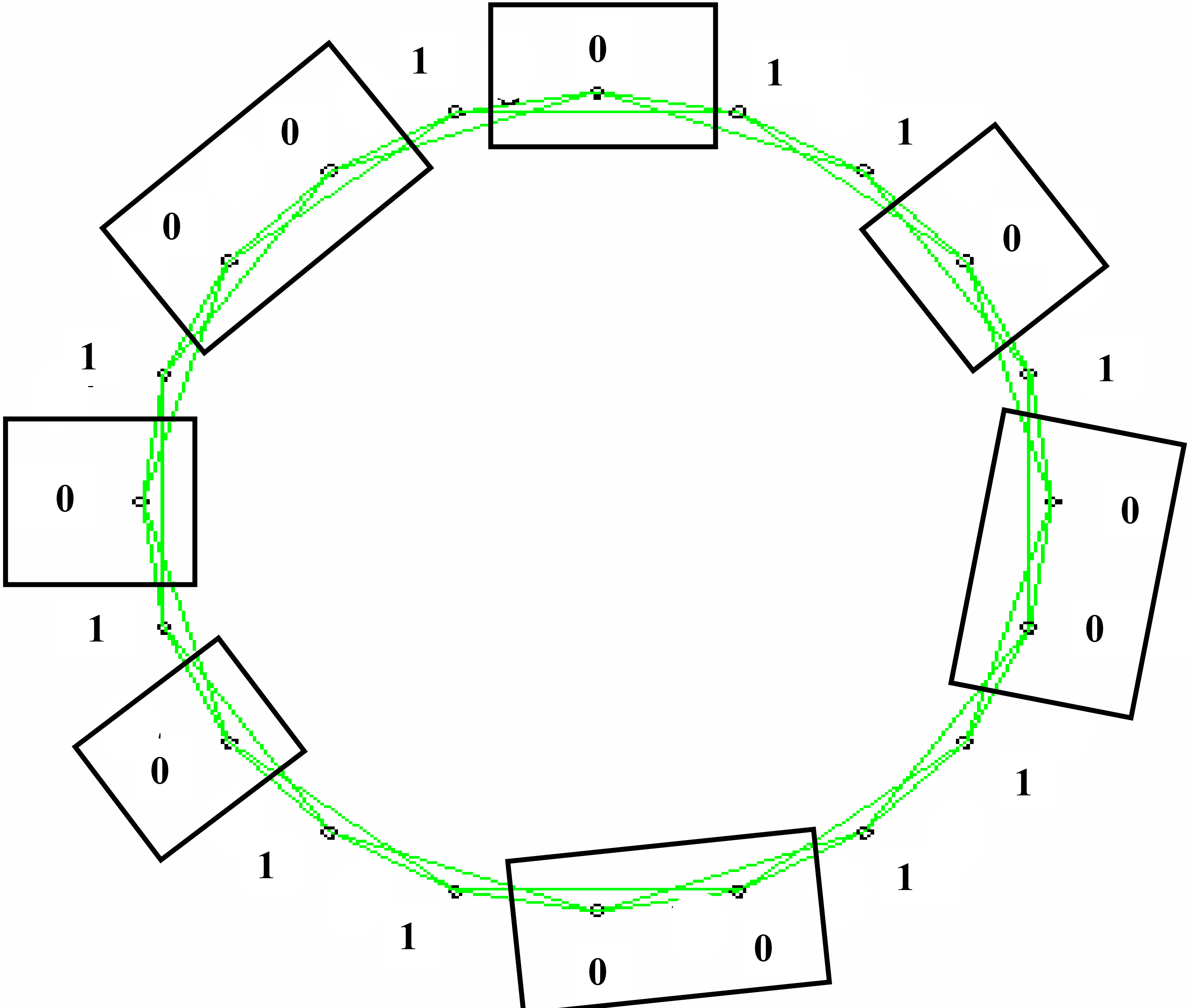
We can view the vertices of a subgraph as the original vertices being labeled by 0s or 1s.

We label a vertex with 1 if it is in the subgraph, and 0 if it is not in the subgraph.

Example:

Third Component

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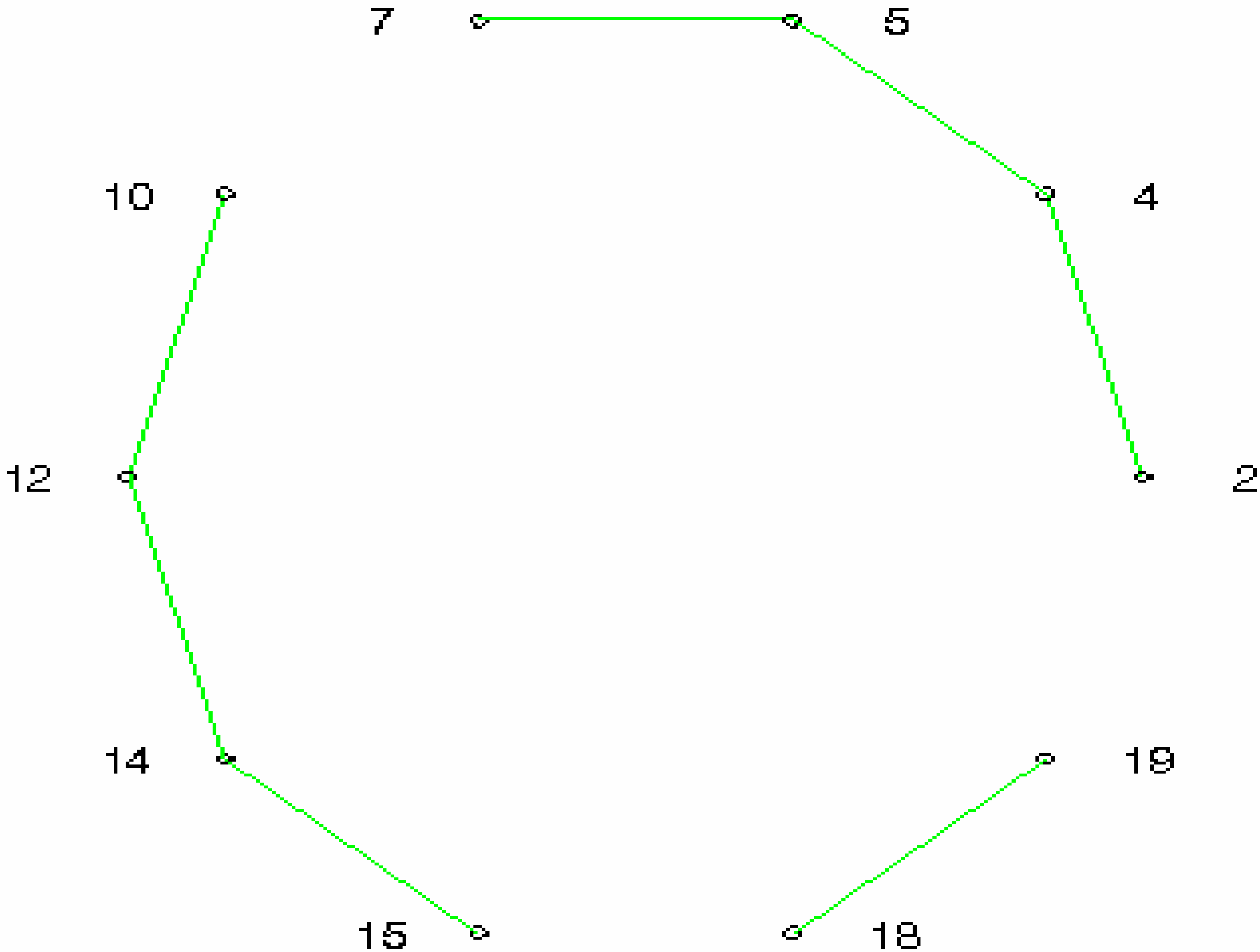
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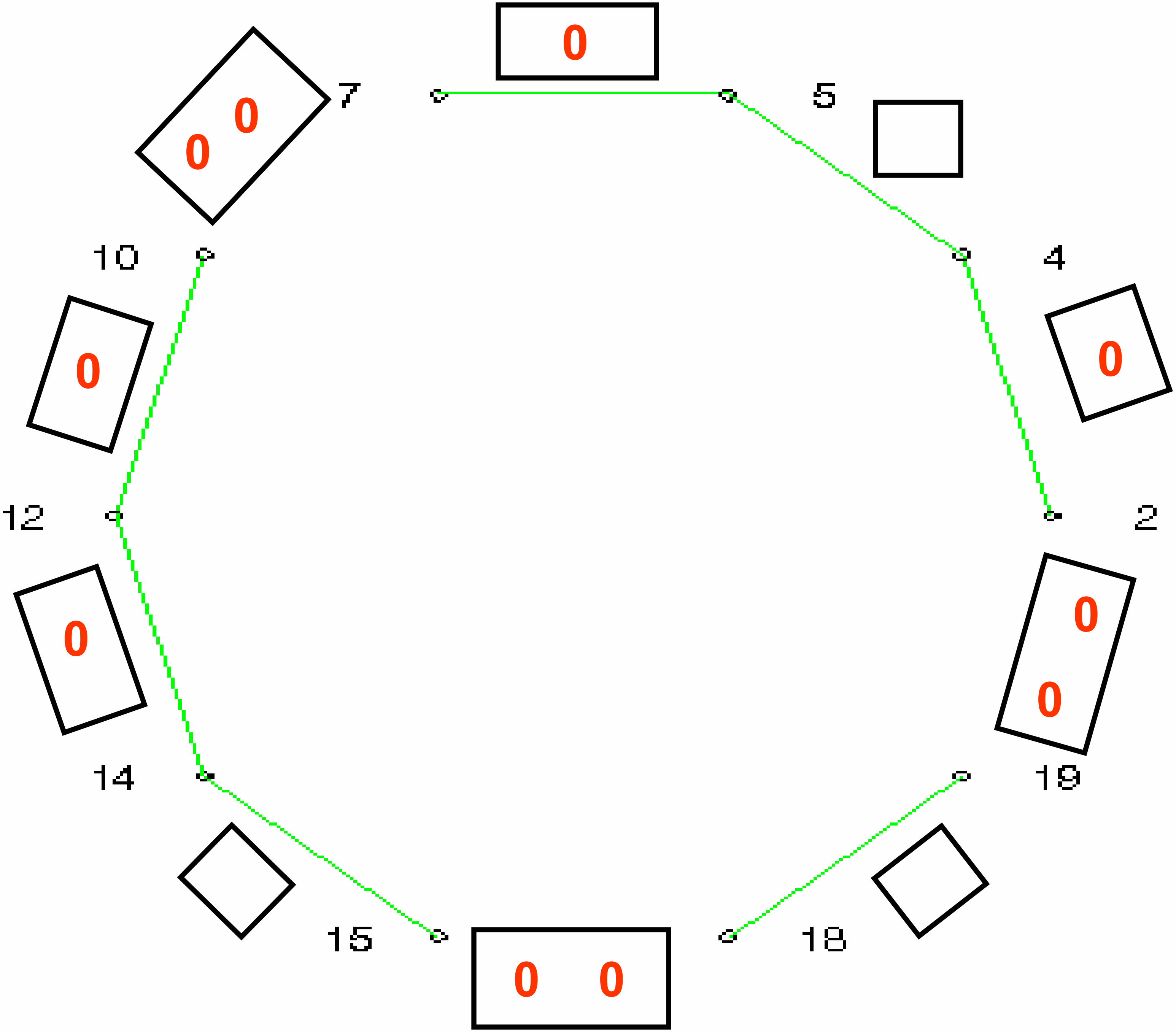

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10
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0
12
2
0
0
0
14
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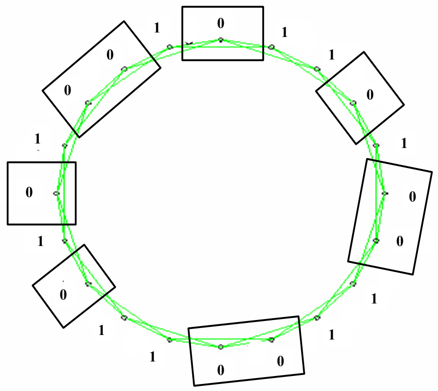
0

n = # of vertices in the original circulant

s = # of 1s (# of vertices in the induced subgraph)

$n - s$ = # of 0s



$Y_{n,s}$ = # of components

X = # of boxes with $\geq k$ objects when $n - s$ objects are uniformly distributed into s boxes

$$X = \sum_{i=1}^{s} X_i \qquad \text{where } X_i = \begin{cases} 0 \text{ if box } i \text{ has } < k \text{ objects} \\ 1 \text{ if box } i \text{ has } \geq k \text{ objects} \end{cases}$$

$$E(X) = \sum_{i=1}^{s} E(X_i)$$

$$E(X_i) = E(X_j) \quad \text{for all } i, j$$

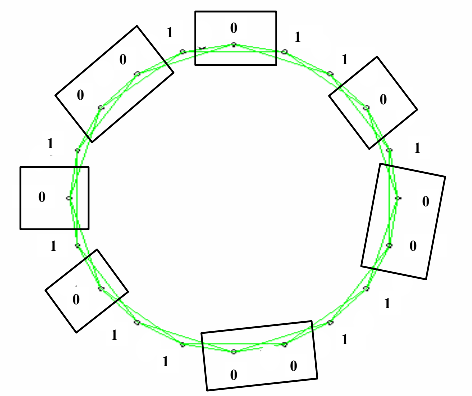
So,

$$E(X) = sE(X_1) = sP(\text{box 1 has} \geq k \text{ objects})$$

n = # of vertices in the original circulant

s = # of 1s (# of vertices in the induced subgraph)

$n - s$ = # of 0s



$Y_{n,s}$ = # of components

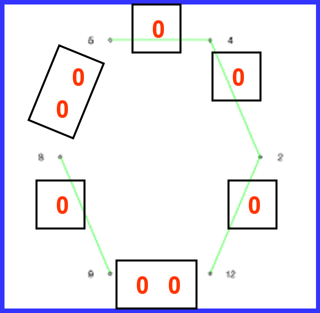
X = # of boxes with $\geq k$ objects when $n - s$ objects are uniformly distributed into s boxes

$$E(Y) = \sum_{j=1}^{\left\lfloor \frac{n-s}{k} \right\rfloor} jP(Y = j)$$

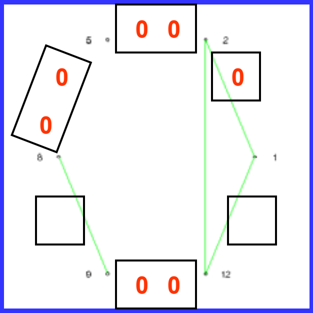
$$E(X) = \sum_{i=0}^{\left\lfloor \frac{n-s}{k} \right\rfloor} iP(X = i) = 0 + \sum_{i=1}^{\left\lfloor \frac{n-s}{k} \right\rfloor} iP(X = i) = \sum_{i=1}^{\left\lfloor \frac{n-s}{k} \right\rfloor} iP(X = i)$$

$$P(Y = j) = P(X = j) \quad \text{for } 2 \leq j \leq \left\lfloor \frac{n-s}{k} \right\rfloor$$

$$P(Y = j) = P(X = 0) + P(X = 1) \quad \text{for } j = 1$$



X = 2
Y = 2

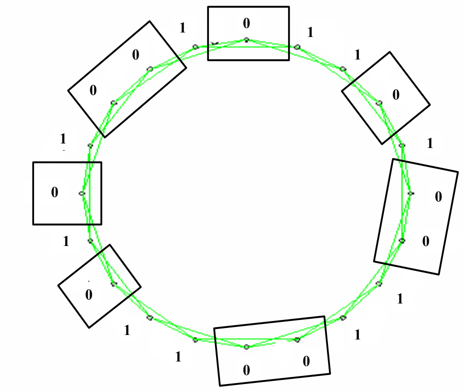


X = 3
Y = 3

n = # of vertices in the original circulant

s = # of 1s (# of vertices in the induced subgraph)

$n - s$ = # of 0s



$Y_{n,s}$ = # of components

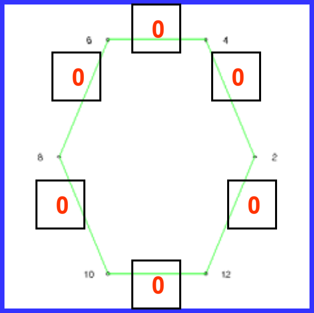
X = # of boxes with $\geq k$ objects when $n - s$ objects are uniformly distribute d into s boxes

$$E(Y) = \sum_{j=1}^{\left\lfloor \frac{n-s}{k} \right\rfloor} jP(Y = j)$$

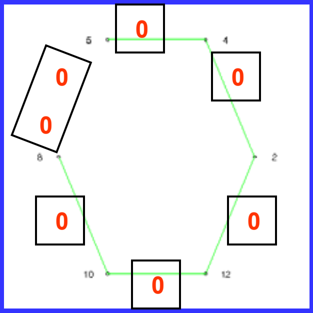
$$E(X) = \sum_{i=0}^{\left\lfloor \frac{n-s}{k} \right\rfloor} iP(X = i) = 0 + \sum_{i=1}^{\left\lfloor \frac{n-s}{k} \right\rfloor} iP(X = i) = \sum_{i=1}^{\left\lfloor \frac{n-s}{k} \right\rfloor} iP(X = i)$$

$$P(Y = j) = P(X = j) \quad \text{for } 2 \leq j \leq \left\lfloor \frac{n-s}{k} \right\rfloor$$

$$P(Y = j) = P(X = 0) + P(X = 1) \quad \text{for } j = 1$$



X = 0
Y = 1

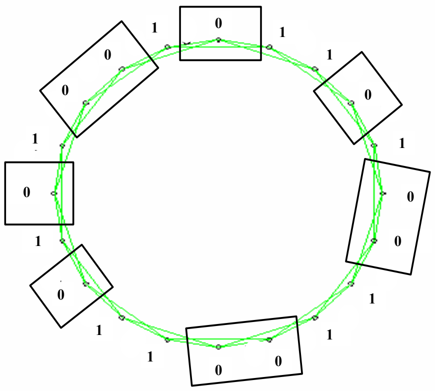


X = 1
Y = 1

n = # of vertices in the original circulant

s = # of 1s (# of vertices in the induced subgraph)

$n - s$ = # of 0s



$Y_{n,s}$ = # of components

X = # of boxes with $\geq k$ objects when $n - s$ objects are uniformly distributed into s boxes

$$E(Y) = \sum_{j=1}^{\left\lfloor \frac{n-s}{k} \right\rfloor} jP(Y = j)$$

$$E(X) = \sum_{i=0}^{\left\lfloor \frac{n-s}{k} \right\rfloor} iP(X = i) = 0 + \sum_{i=1}^{\left\lfloor \frac{n-s}{k} \right\rfloor} iP(X = i) = \sum_{i=1}^{\left\lfloor \frac{n-s}{k} \right\rfloor} iP(X = i)$$

$$P(Y = j) = P(X = j) \quad \text{for } 2 \leq j \leq \left\lfloor \frac{n-s}{k} \right\rfloor$$

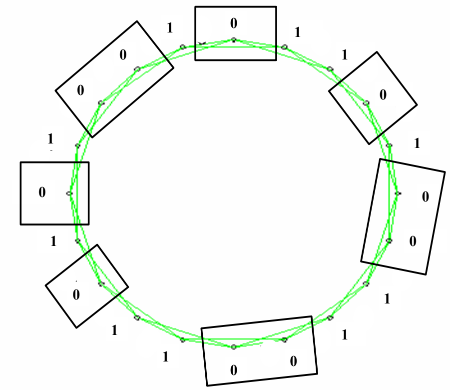
$$P(Y = j) = P(X = 0) + P(X = 1) \quad \text{for } j = 1$$

Therefore...

n = # of vertices in the original circulant

s = # of 1s (# of vertices in the induced subgraph)

$n - s$ = # of 0s



$Y_{n,s}$ = # of components

X = # of boxes with $\geq k$ objects when $n - s$ objects are uniformly distribute d into s boxes

$$E(Y) = P(X = 0) + P(X = 1) + \sum_{i=2}^{\left\lfloor \frac{n-s}{k} \right\rfloor} jP(Y = i)$$

$$= P(X = 0) + P(X = 1) + \sum_{i=2}^{\left\lfloor \frac{n-s}{k} \right\rfloor} iP(X = i)$$

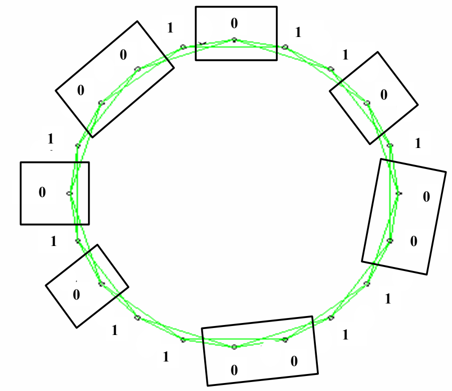
$$= P(X = 0) + E(X)$$

$$= P(X = 0) + sP(\text{box 1 has} \geq k \text{ objects})$$

n = # of vertices in the original circulant

s = # of 1s (# of vertices in the induced subgraph)

$n - s$ = # of 0s



$Y_{n,s}$ = # of components

X = # of boxes with $\geq k$ objects when $n - s$ objects are uniformly distributed into s boxes

$$P(X=0) = \frac{\binom{s+n-s-1}{n-s} - \left|\bigcup_{i=1}^{s} A_i\right|}{\binom{s+n-s-1}{n-s}}$$

where A_i = # of ways to put $n-s$ objects into s boxes, such that box i has $\geq k$ objects

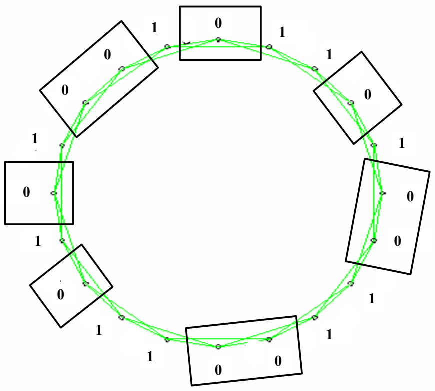
A_i is a regular family, so...

$$\left|\bigcup_{i=1}^{s} A_i\right| = \binom{s}{1}|A_1| - \binom{s}{2}|A_1 \cap A_2| + \binom{s}{3}|A_1 \cap A_2 \cap A_3| - \ldots(-1)^{s-1}\binom{s}{s}|A_1 \cap A_2 \cap \ldots \cap A_s|$$

n = # of vertices in the original circulant

s = # of 1s (# of vertices in the induced subgraph)

$n - s$ = # of 0s



$Y_{n,s}$ = # of components

X = # of boxes with $\geq k$ objects when $n - s$ objects are uniformly distribute d into s boxes

$$P(X=0)=\frac{\binom{n-1}{n-s}-\binom{s}{1}\left|A_1\right|+\binom{s}{2}\left|A_1\cap A_2\right|-\binom{s}{3}\left|A_1\cap A_2\cap A_3\right|+\ldots(-1)^s\binom{s}{s}\left|A_1\cap A_2\cap\ldots\cap A_s\right|}{\binom{n-1}{n-s}}$$

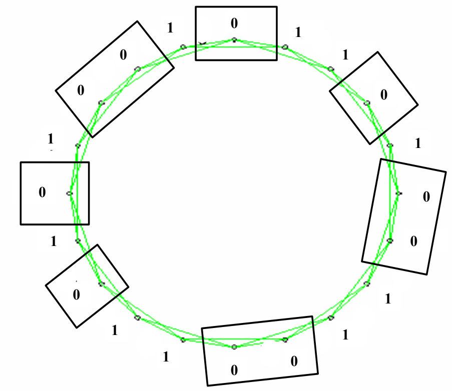
$$E(Y)=\frac{\binom{n-1}{n-s}-\binom{s}{1}\left|A_1\right|+\binom{s}{2}\left|A_1\cap A_2\right|-\binom{s}{3}\left|A_1\cap A_2\cap A_3\right|+\ldots(-1)^s\binom{s}{s}\left|A_1\cap A_2\cap\ldots\cap A_s\right|}{\binom{n-1}{n-s}}$$

$$+\, sP(\text{box 1 has} \geq k \text{ objects})$$

n = # of vertices in the original circulant

s = # of 1s (# of vertices in the induced subgraph)

$n - s$ = # of 0s



$Y_{n,s}$ = # of components

X = # of boxes with $\geq k$ objects when $n - s$ objects are uniformly distributed into s boxes

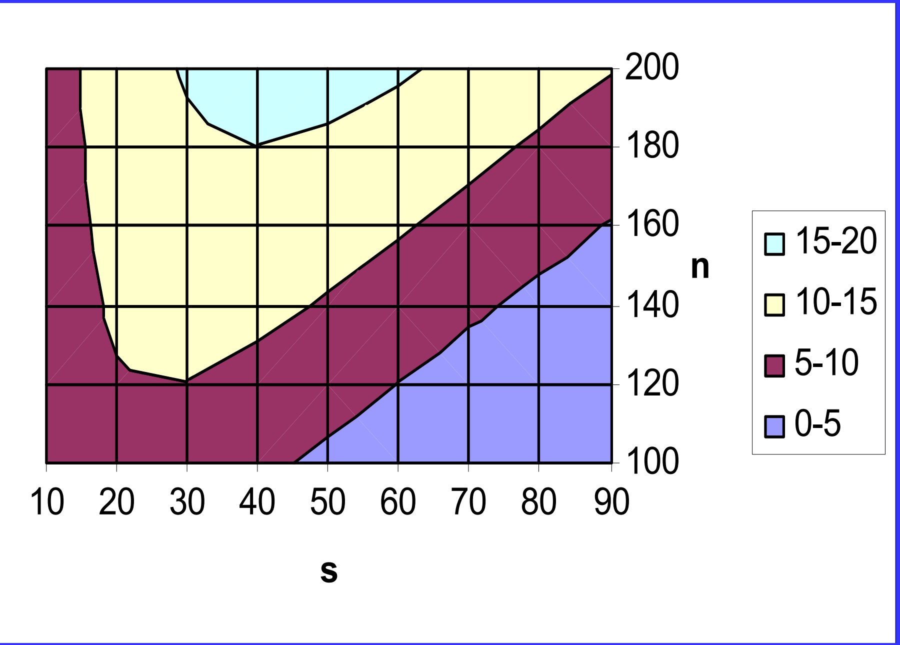
$$P(\text{box 1 has} \geq k \text{ objects}) = \frac{\binom{s+n-s-k-1}{n-s-k}}{\binom{s+n-s-1}{n-s}} = \frac{(n-k-1)!(n-s)!}{(n-1)!(n-s-k)!}$$

Therefore,

$$E(Y) = \frac{\binom{n-1}{n-s} - \binom{s}{1}|A_1| + \binom{s}{2}|A_1 \cap A_2| - \binom{s}{3}|A_1 \cap A_2 \cap A_3| + \ldots(-1)^s \binom{s}{s}|A_1 \cap A_2 \cap \ldots \cap A_s|}{\binom{n-1}{n-s}}$$

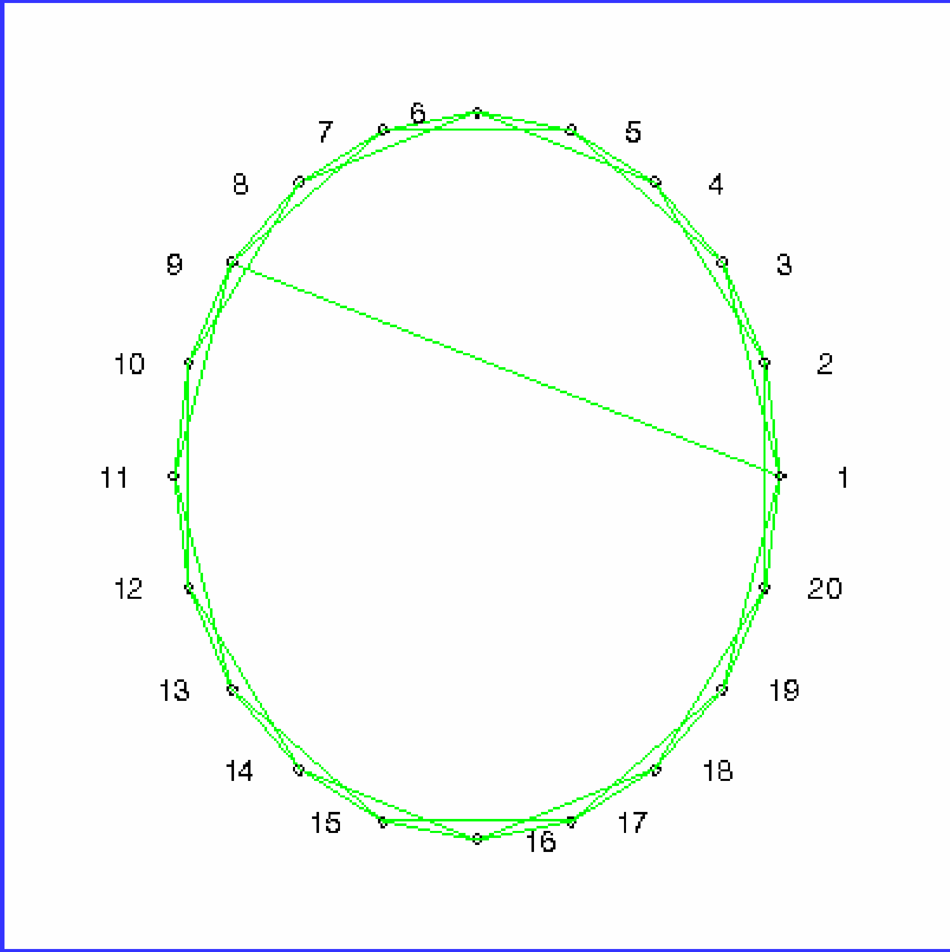
$$+ \frac{s(n-k-1)!(n-s)!}{(n-1)!(n-s-k)!}$$

Expected Number of Components for a 2k regular Circulant with k = 4:

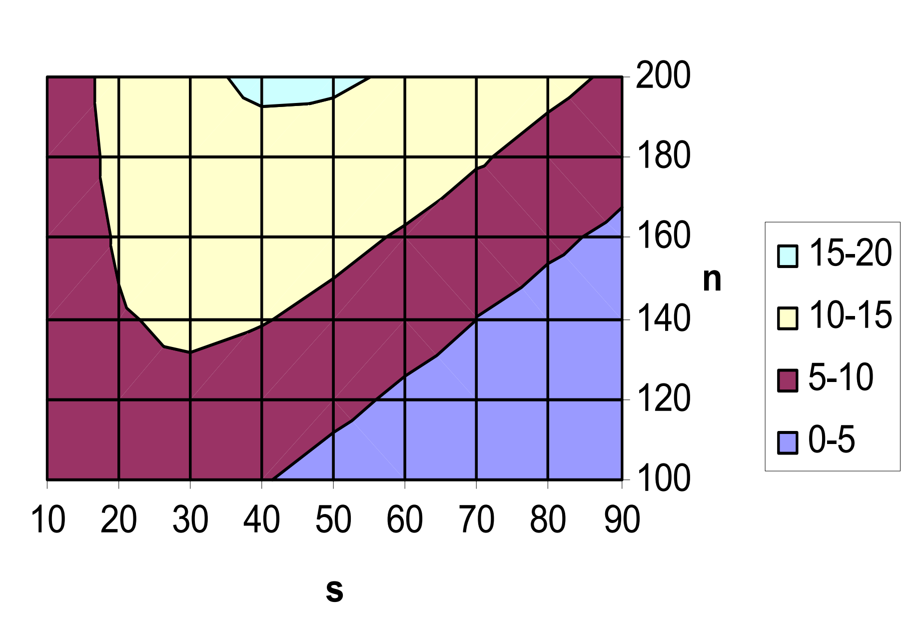


n/s	10	20	30	40	50	60	70	80	90
100	6.39	8.36	7.81	6.08	4.08	2.38	1.37	1.04	1.00
120	6.79	9.63	9.93	8.75	6.89	4.90	3.14	1.87	1.22
140	7.08	10.62	11.67	11.12	9.62	7.70	5.73	3.94	2.51
160	7.31	11.40	13.12	13.17	12.13	10.46	8.51	6.55	4.75
180	7.48	12.03	14.33	14.95	14.40	13.07	11.29	9.33	7.37
200	7.63	12.56	15.35	16.50	16.42	15.47	13.96	12.12	10.15

Cross-Cut Edges



Expected Number of Components for a 2k regular Circulant with k = 4 and 1 added Cross-cut Edge:



n/s	10	20	30	40	50	60	70	80	90
100	5.48	7.46	6.94	5.28	3.43	1.99	1.23	1.01	1.00
120	5.86	8.71	9.02	7.87	6.08	4.20	2.63	1.60	1.13
140	6.15	9.68	10.74	10.20	8.74	6.87	4.98	3.34	2.11
160	6.36	10.45	12.18	12.24	11.22	9.58	7.67	5.77	4.08
180	6.53	11.08	13.38	14.01	13.47	12.15	10.40	8.48	6.57
200	6.67	11.60	14.39	15.55	15.47	14.54	13.04	11.22	9.28